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Structure of contact region for non-symmetric initial disturbances

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SUMMARY

The evolution of small finite non-symmetric initial disturbances governed by the Navier–Stokes equations is considered. Only the contact region is discussed here. The linearized theory is shown to break down and a system of coupled equations is shown to govern the nonlinear contact region. Contrary to the linear theory, the effects of vorticity and entropy are now intertwined. The theory resembles that of an incompressible fluid in two and three space dimensions.

1. Introduction

The work presented here represents a continuation of two earlier studies [1] and [2]. Reference [1] considers the evolution of infinitesimal disturbances of a viscous heat-conducting gas in two and three space dimensions while reference [2] considers the finite amplitude effects, that is, nonlinearities in the case of two and three dimensions for radially symmetric initial data.

In general, initial disturbances resolve themselves into two modes, a signal traveling away from the source of disturbance with a speed related to the local fluid velocity and speed of sound and a relatively slow moving disturbance traveling at the local fluid velocity. The latter mode is called the contact region and carries the vorticity and entropy perturbations.

In this paper, we study the development of small but finite non-symmetric initial disturbances in two and three dimensions. Furthermore we focus our discussion on the contact region. The case of the wave region has been dealt with in [3] and [4]. Whitham [3] considered the evolution of weak shocks for non-symmetric explosions based upon the method of geometrical acoustics. Varley and Cumberbatch [4] studied the general nonlinear theory of wave-front propagation. As an example, they applied their theory to non-symmetric explosions verifying Whitham's results. In both cases, the solutions in the wave region depend upon the principle radii of curvature of the wave front. Thus the wave region will not be discussed further.

In [1], where the linearized theory is dealt with, the contact region is shown to decouple into two parts, an entropy perturbation structured by heat conduction and a vorticity perturbation structured by viscosity. For the case of radially-symmetric initial data [2], only an entropy perturbation is produced. Vorticity is absent from the flow field. The contact region is governed by the two and three dimensional heat equation and dependent upon heat conduction only. However if the initial velocity is non-symmetric, a shearing motion occurs leading to the production of vorticity governed by a diffusion equation initially.

The problem was first looked at by Lagerstrom, Cole and Trilling [5] for linearized theory. In their study, they showed that the linearized Navier–Stokes equations may be split as the sum of a longitudinal wave (wave region) plus a transversal wave (contact region). This was carried out in the absence of heat conduction. The longitudinal waves are irrotational while the transversal waves are incompressible. They show that within the linearized theory, the propagation of vorticity is independent of compressibility. This is again verified by our study.

We restrict attention to the three dimensional equations of motion stated in Section 2. The two dimensional results are given in the appendix for completeness. The one dimensional case has been treated previously [7] and [8]. We wish to find the analogous nonlinear theory to the heat equation. Our approach is based on the method of multiple scales [6]. In Section 3, we develop the linear theory from a different point of view than [1]. If ε represents the smallness parameter, we show that the linear inviscid theory breaks down for $t=O(\varepsilon^{-1})$ at which time the nonlinear terms become important. The nonlinear theory follows as a consequence of the breakdown of linear theory. This derivation is given in the framework of a systematic perturbation procedure and leads to explicit forms for the governing equations. It is shown in Section 5 that a uniformly valid solution to first order is described by a system of two coupled equations. The vorticity is governed by the well known two and three dimensional vorticity equation of incompressible fluids while the density and temperature are described by an equation of similar form which may be solved once the velocity is uncoupled. Contrary to the linear theory, vorticity and entropy are now intertwined.

The theory is of practical interest in explosion problems. For explosions, a very large disturbance of air is desired, thus only the behavior of the theory for large times would be of value once the disturbances have become weak. The problem is interesting mathematically in that the results depend upon a double application of the Fredholm condition.

2. Equations of motion

Introducing the vector notation

$$\hat{\boldsymbol{v}} = \boldsymbol{v}_0 + \boldsymbol{v}$$

where $\boldsymbol{v}_0$ represents the undisturbed flow, the three dimensional normalized equations of motion for the perturbation quantities $\boldsymbol{v}=(\rho, u, v, w, T)$ are

$$\left(\frac{\partial}{\partial t} + \boldsymbol{B}\frac{\partial}{\partial x} + \boldsymbol{C}\frac{\partial}{\partial y} + \boldsymbol{D}\frac{\partial}{\partial z}\right)\boldsymbol{v} = \boldsymbol{X}(\boldsymbol{v}) + \boldsymbol{Y}(\boldsymbol{v}) + \boldsymbol{Z}(\boldsymbol{v}) \tag{1}$$

where

$$\boldsymbol{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \chi \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \chi & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \chi \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \chi & 0 & 0 \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \chi \\ 0 & 0 & 0 & \chi & 0 \end{bmatrix},$$

$$X(v) = -\begin{bmatrix} \rho\dfrac{\partial u}{\partial x} + \rho\dfrac{\partial v}{\partial y} + \rho\dfrac{\partial w}{\partial z} + u\dfrac{\partial \rho}{\partial x} + v\dfrac{\partial \rho}{\partial y} + w\dfrac{\partial \rho}{\partial z} \\ \rho\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} + \chi\rho\dfrac{\partial T}{\partial x} + \chi T\dfrac{\partial \rho}{\partial x} \\ \rho\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} + \chi\rho\dfrac{\partial T}{\partial y} + \chi T\dfrac{\partial \rho}{\partial y} \\ \rho\dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z} + \chi\rho\dfrac{\partial T}{\partial z} + \chi T\dfrac{\partial \rho}{\partial z} \\ \rho\dfrac{\partial T}{\partial t} + u\dfrac{\partial T}{\partial x} + v\dfrac{\partial T}{\partial y} + w\dfrac{\partial T}{\partial z} + \chi\rho\dfrac{\partial u}{\partial x} + \chi\rho\dfrac{\partial v}{\partial y} \\ \qquad + \chi\rho\dfrac{\partial w}{\partial z} + \chi^2 T\dfrac{\partial u}{\partial x} + \chi^2 T\dfrac{\partial v}{\partial y} + \chi^2 T\dfrac{\partial w}{\partial z} \end{bmatrix},$$

$$Y(v) = \begin{bmatrix} 0 \\ (\zeta+\eta)\dfrac{\partial^2 u}{\partial x^2} + \eta\left(\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) + \zeta\dfrac{\partial^2 v}{\partial x \partial y} + \zeta\dfrac{\partial^2 w}{\partial x \partial z} \\ (\zeta+\eta)\dfrac{\partial^2 v}{\partial y^2} + \eta\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial z^2}\right) + \zeta\dfrac{\partial^2 u}{\partial x \partial y} + \zeta\dfrac{\partial^2 w}{\partial y \partial z} \\ (\zeta+\eta)\dfrac{\partial^2 w}{\partial z^2} + \eta\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right) + \zeta\dfrac{\partial^2 u}{\partial x \partial z} + \zeta\dfrac{\partial^2 v}{\partial y \partial z} \\ \xi\left(\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} + \dfrac{\partial^2 T}{\partial z^2}\right) \end{bmatrix},$$

with

$$\zeta = \eta + \lambda = \frac{\beta + (4/3)\mu}{\rho_0 a_0 L}, \quad \eta = \frac{\mu}{\rho_0 a_0 L}, \quad \xi = \frac{\kappa}{\rho_0 c_v a_0 L}.$$

The vector function $\boldsymbol{X}$ represents the quadratic inviscid terms, $\boldsymbol{Y}$ the linear dissipative terms and $\boldsymbol{Z}$ the remaining higher order terms. The dissipative parameters are taken constant, since their variation does not enter in the perturbation procedure introduced later on. The normalization is that used in [1], in particular $\chi = (\gamma-1)^{\frac{1}{2}}$, where γ is the ratio of specific heats and ζ is a reciprocal Reynolds number based on an unspecified length scale L.

3. First order theory—Inviscid

Neglecting dissipation for the moment, we consider the inviscid equations. If ε represents the strength of our initial disturbance, i.e. $\boldsymbol{v}(t=0) = \varepsilon \boldsymbol{v}^0(\boldsymbol{x})$ concentrated in a finite domain, we formally expand

$$\boldsymbol{v} = (\rho, u, v, w, T) = \boldsymbol{v}_0 + \varepsilon \boldsymbol{v}_1 + \varepsilon^2 \boldsymbol{v}_2 + \dots \tag{2}$$

where $\boldsymbol{v}_0 = (1, 0, 0, 0, \chi^{-1})$ and all lengths are normalized with respect to a representative wavelength of the disturbance. From (1), the lowest order equation is

$$\left(\frac{\partial}{\partial t} + \boldsymbol{B}\frac{\partial}{\partial x} + \boldsymbol{C}\frac{\partial}{\partial y} + \boldsymbol{D}\frac{\partial}{\partial z}\right)\boldsymbol{v}_1 = \boldsymbol{0}. \tag{3}$$

Since we are interested in the contact region only, it is convenient to decompose the flow in a way natural to the equations. Introducing the eigenvalue λ^i and eigenvectors $\boldsymbol{l}^i$ of

$$\boldsymbol{I}\lambda^i \boldsymbol{l}^i = 3^{-\frac{1}{2}}(\boldsymbol{B}+\boldsymbol{C}+\boldsymbol{D})\boldsymbol{l}^i \qquad (i = 1, 2, 3, 4, 5)$$

a direction calculation shows

$$\begin{aligned} &\lambda^1 = 0 && \boldsymbol{l}^1 = (\chi, 0, 0, 0, -1) \\ &\lambda^2 = 0 && \boldsymbol{l}^2 = (0, 1, -1, 0, 0) \\ &\lambda^3 = 0 && \boldsymbol{l}^3 = (0, 1, 1, -2, 0) \\ &\lambda^4 = \gamma^{\frac{1}{2}} && \boldsymbol{l}^4 = (1, (\gamma/3)^{\frac{1}{2}}, (\gamma/3)^{\frac{1}{2}}, (\gamma/3)^{\frac{1}{2}}, \chi) \\ &\lambda^5 = -\gamma^{\frac{1}{2}} && \boldsymbol{l}^5 = (1, -(\gamma/3)^{\frac{1}{2}}, -(\gamma/3)^{\frac{1}{2}}, -(\gamma/3)^{\frac{1}{2}}, \chi) \end{aligned} \tag{4}$$

where the $\boldsymbol{l}^i$ are orthogonal, i.e. $\boldsymbol{l}^i \cdot \boldsymbol{l}^j = 0$, $i \neq j$.

The first three vectors can be associated with the contact region while the latter two are associated with the wave region. We further note that $\boldsymbol{l}^2$ and $\boldsymbol{l}^3$ did not occur for radially-symmetric initial data (see [2]).

In general, we seek a solution of the form

$$\boldsymbol{v}_1 = \boldsymbol{l}^1 F_1 + \boldsymbol{l}^2 F_2 + \boldsymbol{l}^3 F_3 + \boldsymbol{l}^4 F_4 + \boldsymbol{l}^5 F_5$$

where

$$F_i = \frac{\boldsymbol{l}^i \cdot \boldsymbol{v}_1}{\boldsymbol{l}^i \cdot \boldsymbol{l}^i} \qquad (i = 1, \ldots, 5).$$

However since we are interested in the contact region only, we look for solutions in the limit $t \to \infty$, $\boldsymbol{x}$ bounded, i.e.

$$\begin{aligned} \boldsymbol{v}_1 \sim\; & t^{-p_1}\{\boldsymbol{l}^1 f_1(x, y, z) + \boldsymbol{l}^2 f_2(x, y, z) + \ldots + \boldsymbol{l}^5 f_5(x, y, z)\} \\ & + t^{-p_2}\{\boldsymbol{l}^1 g_1(x, y, z) + \boldsymbol{l}^2 g_2(x, y, z) + \ldots + \boldsymbol{l}^5 g_5(x, y, z)\} + \ldots \end{aligned} \tag{5}$$

where $0 \leq p_1 < p_2 < \ldots$ and some of the f's and g's may be zero. Substituting into (3) and multiplying from the left by $\boldsymbol{l}^i$ ($i = 1, 2, \ldots, 5$), we obtain

$$i = 1 \qquad 0 = 0$$

$$i = 2 \qquad \gamma \frac{\partial}{\partial x}(f_4 + f_5) = \gamma \frac{\partial}{\partial y}(f_4 + f_5) \tag{6}$$

$$i = 3 \qquad \gamma \frac{\partial}{\partial x}(f_4 + f_5) + \gamma \frac{\partial}{\partial y}(f_4 + f_5) = 2\gamma \frac{\partial}{\partial z}(f_4 + f_5) \tag{7}$$

$$i = 4 \qquad \gamma\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) f_2 + \gamma\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - 2\frac{\partial}{\partial z}\right) f_3 + 2\gamma\left(\frac{\gamma}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) f_4 = 0 \tag{8}$$

$$i = 5 \qquad \gamma\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) f_2 + \gamma\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - 2\frac{\partial}{\partial z}\right) f_3 - 2\gamma\left(\frac{\gamma}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) f_5 = 0 \tag{9}$$

To analyse the system (6)–(9), we restrict attention to solutions which vanish as $x^2 + y^2 + z^2 \to \infty$ or more strongly from energy considerations, we may take only square integrable functions. Subtracting (9) from (8) and combining with (6) and (7), we find that

$$f_4(x, y, z) = -f_5(x, y, z). \tag{10}$$

Using (10), the above system of equations reduces to (8) or written alternatively as

$$\frac{\partial}{\partial x}\left(f_2 + f_3 + 2\left(\frac{\gamma}{3}\right)^{\frac{1}{2}} f_4\right) + \frac{\partial}{\partial y}\left(f_3 - f_2 + 2\left(\frac{\gamma}{3}\right)^{\frac{1}{2}} f_4\right) + \frac{\partial}{\partial z}\left(-2f_3 + 2\left(\frac{\gamma}{3}\right)^{\frac{1}{2}} f_4\right) = 0.$$

This takes the form of a divergence, i.e.

$$\nabla \cdot \boldsymbol{\psi} = 0 \text{ where } \boldsymbol{\psi} = (\psi^1, \psi^2, \psi^3) \tag{11}$$

and

$$\psi^1 = f_2 + f_3 + 2\left(\frac{\gamma}{3}\right)^{\frac{1}{2}} f_4, \quad \psi^2 = f_3 - f_2 + 2\left(\frac{\gamma}{3}\right)^{\frac{1}{2}} f_4, \quad \psi^3 = -2f_3 + 2\left(\frac{\gamma}{3}\right)^{\frac{1}{2}} f_4.$$

It will be shown later that these are the components of velocity in the contact region. Equation (11) implies the existence of a vector potential

$$\boldsymbol{\psi} = \nabla \times \boldsymbol{A}, \qquad \boldsymbol{A} = (A_1, A_2, A_3).$$

It can be shown that

$$f_2 = \frac{1}{2}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} - \frac{\partial A_1}{\partial z} + \frac{\partial A_3}{\partial x}\right) = \tfrac{1}{2}(\psi^1 - \psi^2),$$

$$f_3 = \frac{1}{6}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} + \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} - 2\frac{\partial A_2}{\partial x} + 2\frac{\partial A_1}{\partial y}\right) = \tfrac{1}{6}(\psi^1 + \psi^2 - 2\psi^3),$$

$$\begin{aligned} f_4 = -f_5 &= \frac{1}{6(\gamma/3)^{\frac{1}{2}}}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} + \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} + \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) \\ &= \frac{1}{6(\gamma/3)^{\frac{1}{2}}}(\psi^1 + \psi^2 + \psi^3). \end{aligned} \tag{12}$$

We can now rewrite (5) as

$$v_1 = t^{-p_1}\{l^1 f_1 + \psi^1 l^x + \psi^2 l^y + \psi^3 l^z\} + O(t^{-p_2}), \tag{13}$$

where $l^x = (0, 1, 0, 0, 0)$, $l^y = (0, 0, 1, 0, 0)$ and $l^z = (0, 0, 0, 1, 0)$. It is at this point that ψ is recognized as just the velocity in the contact region. Thus we have reduced the solution to four arbitrary functions $f_1, \psi^1, \psi^2, \psi^3$ which may be obtained from the fundamental solution of the linearized Euler equations (see [1]).

We make the observation from (11) that the vorticity part of the contact region is basically incompressible. This was observed earlier by Lagerstrom, Cole and Trilling [5], that is within the linearized theory, the propagation of vorticity is independent of compressibility.

To determine p_1, we consider the second order equation found by substituting (5) into (3). Setting $p_2 = p_1 + 1$, we have

$$-p_1\{l^1 f_1 + \psi^1 l^x + \psi^2 l^y + \psi^3 l^z\} + \left\{B\frac{\partial}{\partial x} + C\frac{\partial}{\partial y} + D\frac{\partial}{\partial z}\right\}\{l^1 g_1 + l^2 g_2 + \ldots + l^5 g_5\} = 0 .$$

Multiplying from the left by $l^{1,x,y,z}$ respectively, we find that for a nontrivial lowest order solution, $p_1 = 0$. The g's will then satisfy equations of the form (6)–(9). Thus the lowest order solution is

$$v_1 \sim l^1 f_1(x, y, z) + \psi^1(x, y, z) l^x + \psi^2(x, y, z) l^y + \psi^3(x, y, z) l^z .$$

It is clear that succeeding orders will depend on solutions of the nonhomogeneous form of (6)–(9). Before going into the detailed solution, we first consider this set of four equations and write them symbolically as

$$\gamma L w = G , \tag{14}$$

where L is the symmetric linear operator

$$L = \begin{bmatrix} 0 & 0 & \partial_x - \partial_y & \partial_x - \partial_y \\ 0 & 0 & \partial_x + \partial_y - 2\partial_z & \partial_x + \partial_y - 2\partial_z \\ \partial_x - \partial_y & \partial_x + \partial_y - 2\partial_z & 2(\gamma/3)^{\frac{1}{2}}(\partial_x + \partial_y + \partial_z) & 0 \\ \partial_x - \partial_y & \partial_x + \partial_y - 2\partial_z & 0 & -2(\gamma/3)^{\frac{1}{2}}(\partial_x + \partial_y + \partial_z) \end{bmatrix}$$

and w and G are both four vectors. (We have suppressed the l^1 component of v for the moment since it plays no role in this current discussion.) We recall that for $G = 0$, it was demonstrated that $Lw = 0$ has a nontrivial solution representable in terms of A_i or equivalently ψ^i $(i = 1, 2, 3)$. This suggests that (14) can be solved if and only if the Fredholm condition is met. To be more specific, let us consider the inner product defined by

$$(a, b) = \iiint_{-\infty}^{\infty} (a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4)\,dx\,dy\,dz ,$$

where $a = (a_1, a_2, a_3, a_4)$ and $b = (b_1, b_2, b_3, b_4)$. Since we are only considering square-integrable functions, it follows that

$$(c, Lw) = -(w, Lc) = (c, G) .$$

Therefore the condition of solvability of (14) is that

$$(c, G) = 0 , \tag{15}$$

where c is an element of the subspace of solutions of $Lw = 0$. In fact, we have demonstrated that such an c exists in the form given by (12).

Hence condition (15) for $G = (G_1, G_2, G_3, G_4)$ may be shown to be

$$\iiint_{-\infty}^{\infty}\Bigg[A_1\Bigg\{-\frac{1}{2}\frac{\partial G_1}{\partial z}+\frac{1}{6}\frac{\partial G_2}{\partial z}+\frac{1}{3}\frac{\partial G_2}{\partial y}+\frac{1}{6(\gamma/3)^{\frac{1}{2}}}\frac{\partial G_3}{\partial z}$$
$$-\frac{1}{6(\gamma/3)^{\frac{1}{2}}}\frac{\partial G_3}{\partial y}-\frac{1}{6(\gamma/3)^{\frac{1}{2}}}\frac{\partial G_4}{\partial z}+\frac{1}{6(\gamma/3)^{\frac{1}{2}}}\frac{\partial G_4}{\partial y}\Bigg\}$$
$$+A_2\Bigg\{-\frac{1}{2}\frac{\partial G_1}{\partial z}-\frac{1}{6}\frac{\partial G_2}{\partial z}-\frac{1}{3}\frac{\partial G_2}{\partial x}-\frac{1}{6(\gamma/3)^{\frac{1}{2}}}\frac{\partial G_3}{\partial z}$$
$$+\frac{1}{6(\gamma/3)^{\frac{1}{2}}}\frac{\partial G_3}{\partial x}+\frac{1}{6(\gamma/3)^{\frac{1}{2}}}\frac{\partial G_4}{\partial z}-\frac{1}{6(\gamma/3)^{\frac{1}{2}}}\frac{\partial G_4}{\partial z}\Bigg\}$$
$$+A_3\Bigg\{\frac{1}{2}\frac{\partial G_1}{\partial y}+\frac{1}{2}\frac{\partial G_1}{\partial x}+\frac{1}{6}\frac{\partial G_2}{\partial y}-\frac{1}{6}\frac{\partial G_2}{\partial x}$$
$$+\frac{1}{6(\gamma/3)^{\frac{1}{2}}}\frac{\partial G_3}{\partial y}-\frac{1}{6(\gamma/3)^{\frac{1}{2}}}\frac{\partial G_3}{\partial x}-\frac{1}{6(\gamma/3)^{\frac{1}{2}}}\frac{\partial G_4}{\partial y}+\frac{1}{6(\gamma/3)^{\frac{1}{2}}}\frac{\partial G_4}{\partial x}\Bigg\}\Bigg]dx\,dy\,dz=0$$

after an integration by parts and a regrouping of terms. Since A_1, A_2, A_3 are arbitrary scalar functions, we must have that the integrands inside curly brackets vanish identically or

$$\begin{aligned}
&\partial_z G_1-\tfrac{1}{3}(2\partial_y+\partial_z)G_2-\frac{1}{3(\gamma/3)^{\frac{1}{2}}}(\partial_z-\partial_y)(G_3-G_4)=0\,,\\
&\partial_z G_1+\tfrac{1}{3}(2\partial_x+\partial_z)G_2-\frac{1}{3(\gamma/3)^{\frac{1}{2}}}(\partial_x-\partial_z)(G_3-G_4)=0\,,\\
&(\partial_y+\partial_z)G_1+\tfrac{1}{3}(\partial_y-\partial_x)G_2+\frac{1}{3(\gamma/3)^{\frac{1}{2}}}(\partial_y-\partial_x)(G_3-G_4)=0\,.
\end{aligned}\tag{16}$$

Thus in order to solve the nonhomogeneous equations (14), $\boldsymbol{G}$ must satisfy the above equations. At this point we see that the restriction of square-integrable functions can be relaxed. In fact (16) only requires $\boldsymbol{G}$ to be differentiable.

4. Second order theory—Inviscid

We regard the expansion of $\boldsymbol{v}_1$ as known and return to the solution of the inviscid form of (1) under the expansion (2). The second order equation obtained by substituting (2) into (1) is

$$\left(\frac{\partial}{\partial t}+\boldsymbol{B}\frac{\partial}{\partial x}+\boldsymbol{C}\frac{\partial}{\partial y}+\boldsymbol{D}\frac{\partial}{\partial z}\right)\boldsymbol{v}_2=\boldsymbol{X}(\boldsymbol{v}_1)\tag{17}$$

in which it is to be recalled that $\boldsymbol{X}$ is quadratic. The solution $\boldsymbol{v}_2$ will consist of a homogeneous solution of the same form as $\boldsymbol{v}_1$ plus a particular solution.

We expand $\boldsymbol{v}_2$ in the same form as $\boldsymbol{v}_1$, i.e.

$$\boldsymbol{v}_2=\boldsymbol{l}^1H_1(x,y,z,t)+\boldsymbol{l}^2H_2(x,y,z,t)+\ldots+\boldsymbol{l}^5H_5(x,y,z,t)\,.\tag{18}$$

Substituting (18) into (17) and multiplying from the left by $\boldsymbol{l}^1$ yields

$$H_1(x,y,z,t)=\frac{1}{\gamma}(\boldsymbol{l}^1\cdot\boldsymbol{X}(\boldsymbol{v}_1))t\,,$$

which gives rise to secularity as $t\to\infty$. For $\boldsymbol{l}^i$ $(i=2,3,4,5)$, we obtain

$$\begin{bmatrix}2&&&\\&6&&\\&&2\gamma&\\&&&2\gamma\end{bmatrix}\frac{\partial}{\partial t}\boldsymbol{H}+\gamma\boldsymbol{L}\boldsymbol{H}=\begin{bmatrix}\boldsymbol{l}^2\cdot\boldsymbol{X}\\\boldsymbol{l}^3\cdot\boldsymbol{X}\\\boldsymbol{l}^4\cdot\boldsymbol{X}\\\boldsymbol{l}^5\cdot\boldsymbol{X}\end{bmatrix}\tag{19}$$

We attempt to solve (19) by expanding $\boldsymbol{H}=(H_2, H_3, H_4, H_5)$ for large t as

$$\boldsymbol{H} = \boldsymbol{h}(x, y, z)+o(1), \tag{20}$$

i.e. assuming time independent to lowest order. Then to lowest order, (19) gives

$$\gamma \boldsymbol{L h} = \boldsymbol{G}, \tag{21}$$

where

$$\boldsymbol{h} = (h_2, h_3, h_4, h_5) \quad \text{and} \quad \boldsymbol{G} = (\boldsymbol{l}^2\cdot\boldsymbol{X}, \boldsymbol{l}^3\cdot\boldsymbol{X}, \boldsymbol{l}^4\cdot\boldsymbol{X}, \boldsymbol{l}^5\cdot\boldsymbol{X}).$$

However, (21) has no solution because the solvability conditions (16) cannot be met. We therefore rewrite $\boldsymbol{H}$ as

$$\boldsymbol{H} = t\,\boldsymbol{h}(x, y, z)+\boldsymbol{k}(x, y, z)+o(1) \tag{22}$$

as $t\to\infty$. After substituting (22) into (19) we can rewrite the equations symbolically as

$$\gamma t \boldsymbol{L h}+\gamma \boldsymbol{L k} = \boldsymbol{G}+\boldsymbol{M}, \tag{23}$$

where $\boldsymbol{M}=(-2h_2, -6h_3, -2\gamma h_4, -2\gamma h_5)$ and $\boldsymbol{G}$ is as above. The lowest order equation from (23) is $\boldsymbol{L h}=\boldsymbol{0}$ which we showed earlier (see (12)) has solutions of the form

$$h_2 = \tfrac{1}{2}(\phi^1-\phi^2), \; h_3 = \tfrac{1}{6}(\phi^1+\phi^2-2\phi^3), \; h_4 = -h_5 = 6(\gamma/3)^{-\frac{1}{2}}(\phi^1+\phi^2+\phi^3),$$

with $\boldsymbol{\phi}=\nabla\times\hat{\boldsymbol{A}}$ undetermined. At the next order

$$\gamma \boldsymbol{L k} = \boldsymbol{M}+\boldsymbol{G}. \tag{24}$$

Equation (24) is not solvable unless

$$(\boldsymbol{s}, \boldsymbol{M}+\boldsymbol{G}) = 0, \tag{25}$$

where $\boldsymbol{s}$ is a nontrivial solution of $\boldsymbol{L s}=\boldsymbol{0}$. The compatability condition (25) is equivalent to (16) and thus gives a relation between $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ of $\boldsymbol{v}_1$ which is assumed known. This then permits a solution to (24) and the process may be continued.

The main conclusion of the above is that secularity appears at the second order since

$$\boldsymbol{v} = \boldsymbol{v}_0+\varepsilon O(1)+\varepsilon^2 O(t)+\dots$$

as $t\to\infty$. Hence it indicates that a new scale $\tau=\varepsilon t$ will be required.

5. Method of multiple scales—Nonlinear description

To overcome the breakdown of linear theory for large times, we employ the method of multiple scales [6]. A solution is sought in the form

$$\boldsymbol{v} = \boldsymbol{v}_0+\varepsilon\boldsymbol{v}_1+\mu(\varepsilon)\boldsymbol{v}_2+\dots, \tag{26}$$

where $\mu(\varepsilon)=o(\varepsilon)$ as $\varepsilon\to 0$, and

$$\boldsymbol{v}_i = \boldsymbol{v}_i(x, y, z, t; \tau),$$

where x, y, z, t are fast variables and $\tau=\varepsilon t$ is a slow scale.

The region of interest is now specified by the conditions $\boldsymbol{x}$, τ fixed with $\varepsilon\to 0$. Substituting (26) into (1), the lowest order equation is again (3) and the solution is given by

$$\boldsymbol{v}_1 \sim \boldsymbol{l}^1 f_1(\boldsymbol{x};\tau)+\boldsymbol{l}^x\psi^1(\boldsymbol{x};\tau)+\boldsymbol{l}^y\psi^2(\boldsymbol{x};\tau)+\boldsymbol{l}^z\psi^3(\boldsymbol{x};\tau),$$

where $\boldsymbol{\psi}=\nabla\times\boldsymbol{A}$. As indicated, the slow dependence on τ is still unknown. At the next order, we obtain

$$\mu(\varepsilon)\left[\frac{\partial}{\partial t}+\boldsymbol{B}\frac{\partial}{\partial x}+\boldsymbol{C}\frac{\partial}{\partial y}+\boldsymbol{D}\frac{\partial}{\partial z}\right]\boldsymbol{v}_2 = \varepsilon^2\left[-\frac{\partial \boldsymbol{v}_1}{\partial\tau}+\boldsymbol{X}(\boldsymbol{v}_1)\right]+\varepsilon\boldsymbol{Y}(\boldsymbol{v}_1). \tag{27}$$

We point out that $Y(v_1)$ is usually small except in regions of high shear where it can be shown that all terms on the right-hand side of (27) are comparable. This involves rescaling the problem with respect to the Reynolds number R, i.e. $\tau = v(\varepsilon, t, R)$. Rather than doing this, we attempt a uniform description by retaining all the leading terms in passing from the inviscid through dissipative zones. See [2] for further comments.

Setting both sides of (27) to the same order, we have $\mu(\varepsilon) = \varepsilon^2$. Next decompose v_2 into its characteristic modes

$$v_2 = l^1 H_1 + l^2 H_2 + l^3 H_3 + l^4 H_4 + l^5 H_5 \,. \tag{28}$$

Substitute (28) into (27) and multiply by l^i $(i=1, \ldots, 5)$ from the left. For l^1, we obtain

$$\gamma \frac{\partial H_1}{\partial t} = l^1 \cdot [-\partial_\tau v_1 + X(v_1) + \varepsilon^{-1} Y(v_1)] \,. \tag{29}$$

Since the right hand side of (29) is independent of t, we can integrate directly to obtain

$$H_1 = O(t) \text{ as } t \to \infty \,.$$

To suppress this secularity, we set the right-hand side of (29) equal to zero. This can be shown to reduce to

$$\gamma \frac{\partial f_1}{\partial \tau} + \gamma \psi \cdot \nabla f_1 = \varepsilon^{-1} \xi \nabla^2 f_1 \,.$$

For $i = 2, 3, 4, 5$, we obtain the following set of equations

$$\begin{bmatrix} 2 \\ 6 \\ 2\gamma \\ 2\gamma \end{bmatrix} \frac{\partial}{\partial t} H + \gamma L H = \begin{bmatrix} l^2 \cdot \\ l^3 \cdot \\ l^4 \cdot \\ l^5 \cdot \end{bmatrix} \left[-\frac{\partial v_1}{\partial \tau} + X + \varepsilon^{-1} Y \right] . \tag{30}$$

From calculations of Section 4, we note that secularity appears if we assume an expansion for H in the form (22). To avoid this difficulty, we would like to expand H in the form (20), i.e.

$$H(x, t; \tau) = h(x) + o(1) \text{ as } t \to \infty \,.$$

Then (30) reduces to $\gamma L h = N$. In order for a solution to exist, we now utilize the dependence of N on τ to impose the solvability conditions (16). The result is

$$\frac{\partial}{\partial \tau} (\nabla \times \psi) + \nabla \times \{(\psi \cdot \nabla)\psi\} = \varepsilon^{-1} \eta \nabla \times (\nabla^2 \psi) \,. \tag{31}$$

Defining the vorticity vector by

$$\omega = \nabla \times \psi \,,$$

(31) can be rewritten as

$$\frac{\partial \omega}{\partial \tau} + (\psi \cdot \nabla)\omega - (\omega \cdot \nabla)\psi = \varepsilon^{-1} \eta \nabla^2 \omega \,,$$

which is just the three-dimensional vorticity equation.

Regarding ε simply as a formal small parameter now and eliminating it by setting $\varepsilon = 1$, we conclude that the nonlinear description of the contact region for a non-symmetric initial disturbance is governed by

$$\frac{\partial f_1}{\partial t} + \psi \cdot \nabla f_1 = \frac{\xi}{\gamma} \nabla^2 f_1 \tag{32}$$

and

$$\frac{\partial \omega}{\partial t} + (\psi \cdot \nabla)\omega - (\omega \cdot \nabla)\psi = \eta \nabla^2 \omega \,. \tag{33}$$

Equations (32) and (33) provide a uniformly valid first order description of the contact region. The contact region is governed by a system of coupled equations. Contrary to the linear theory, the effects of viscosity and heat conduction are now intertwined. If the velocity uncouples from the vorticity equation (33), then density and temperature may be solved from (32) once ψ is known.

We further note that if the initial velocity is zero or in the radial direction only (i.e., $v=v(r)$), no vorticity is produced and $\psi\equiv 0$ in the contact region so that the above equations reduce to

$$\frac{\partial f_1}{\partial t} = \frac{\xi}{\gamma}\nabla^2 f_1\,,$$

which is the linearized result found in [1] and [2]. The solution is given in [1] so no further discussion is necessary.

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The work reported here was obtained in the course of research sponsored by the National Research Council of Canada.

Appendix

For the sake of completeness, we outline the two-dimensional calculations below. Suppressing w, z and letting $\boldsymbol{v}=(\rho, u, v, T)$ denote perturbations, the equations of motion are

$$\left(\frac{\partial}{\partial t} + \boldsymbol{B}\frac{\partial}{\partial x} + \boldsymbol{C}\frac{\partial}{\partial y}\right)\boldsymbol{v} = \boldsymbol{X}(\boldsymbol{v})+\boldsymbol{Y}(\boldsymbol{v})+\boldsymbol{Z}(\boldsymbol{v})\,.$$

Letting

$$\boldsymbol{v} = \boldsymbol{v}_0+\varepsilon\boldsymbol{v}_1+\varepsilon^2\boldsymbol{v}_2+\dots\,,$$

the lowest order equation is

$$\left[\frac{\partial}{\partial t} + \boldsymbol{B}\frac{\partial}{\partial x} + \boldsymbol{C}\frac{\partial}{\partial y}\right]\boldsymbol{v}_1 = \boldsymbol{0}\,.$$

Denoting the eigenvalues and eigenvectors by

$$\boldsymbol{I}\lambda^i\boldsymbol{l}^i = 2^{-\frac{1}{2}}(\boldsymbol{B}+\boldsymbol{C})\boldsymbol{l}^i\,,$$

we find

$$\begin{array}{ll}
\lambda^1 = 0 & \boldsymbol{l}^1 = (\chi, 0, 0, -1)\\
\lambda^2 = 0 & \boldsymbol{l}^2 = (0, 1, -1, 0)\\
\lambda^3 = \gamma^{\frac{1}{2}} & \boldsymbol{l}^3 = (1, (\gamma/2)^{\frac{1}{2}}, (\gamma/2)^{\frac{1}{2}}, \chi)\\
\lambda^4 = -\gamma^{\frac{1}{2}} & \boldsymbol{l}^4 = (1, -(\gamma/2)^{\frac{1}{2}}, -(\gamma/2)^{\frac{1}{2}}, \chi)
\end{array}$$

where again $\boldsymbol{l}^i\cdot\boldsymbol{l}^j=0$ $(i\neq j)$.

Following the same procedure as before, it can be shown that

$$\boldsymbol{v}_1 = \left\{\boldsymbol{l}^1 f_1(x, y)+\boldsymbol{l}^x\frac{\partial\psi}{\partial y} - \boldsymbol{l}^y\frac{\partial\psi}{\partial x}\right\} + O(t^{-1})\,,$$

where ψ is a streamfunction.

The condition for solvability of $\gamma\boldsymbol{L}\boldsymbol{w}=\boldsymbol{G}$ is that $\boldsymbol{G}$ satisfy

$$(\partial_x+\partial_y)G_1 + \frac{1}{2}\left(\frac{\gamma}{2}\right)^{\frac{1}{2}}(\partial_y-\partial_x)(G_2-G_3) = 0\,.$$

Carrying out the procedure identical to Section 4, we find that secularity appears at the second order, i.e.

$$v = v_0 + \varepsilon O(1) + \varepsilon^2 O(t) \text{ as } t \to \infty .$$

Hence again, we use the method of multiple scales where $\tau = \varepsilon t$. Suppressing the secularities and setting $\varepsilon = 1$, we find that

$$\frac{\partial f_1}{\partial t} + (w \cdot \nabla) f_1 = \frac{\xi}{\gamma} \nabla^2 f_1 ,$$

$$\frac{\partial}{\partial t} (\nabla^2 \psi) + (w \cdot \nabla) \nabla^2 \psi = \eta \nabla^2 \psi ,$$

where $w = (\psi_y, -\psi_x)$ and $\nabla = (\partial_x, \partial_y)$.

If we define a vorticity vector by $\boldsymbol{\omega} = (0, 0, \omega)$ and set $\omega = \nabla \times w$, we find $\boldsymbol{\omega} = -\nabla^2 \psi$. Hence our equations may be rewritten as

$$\frac{\partial f_1}{\partial t} + (w \cdot \nabla) f_1 = \xi / \gamma \nabla^2 f_1$$

and

$$\frac{\partial \omega}{\partial t} + (w \cdot \nabla) \omega = \eta \nabla^2 \omega .$$

The latter is precisely the two-dimensional vorticity equation.

REFERENCES

[1] L. Halabisky and L. Sirovich, On the structure of dissipative waves in two and three dimensions, *Quart. Appl. Math.*, 31 (1971) 135–149.

[2] L. Halabisky and L. Sirovich, Evolution of finite disturbances in dissipative gasdynamics, Part II (accepted for publication, *Physics of Fluids*).

[3] G. B. Whitham, On the propagation of weak shock waves, *Journal of Fluid Mechanics*, 1 (1956) 290–318.

[4] E. Varley and E. Cumberbatch, Nonlinear theory of wave-front propagation, *J. Inst. Maths. Applics.*, 1 (1965) 101–112.

[5] P. A. Lagerstrom, J. D. Cole and L. Trilling, California Institute of Technology, Guggenheim Aeronautical Laboratory Report (1949).

[6] J. D. Cole, *Perturbation Methods in Applied Mathematics*, (Blaisdell, Waltham, Mass., 1968) Chapt. 2.

[7] L. Sirovich, Initial and boundary value problems in dissipative gasdynamics, *Physics of Fluids*, 10 (1967) 24–34.

[8] L. Halabisky and L. Sirovich, Evolution of finite disturbances in dissipative gasdynamics, *Physics of Fluids*, 16 (1973) 360–368.